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Aerodynamic sound generation in a pipe

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The paper deals with the problem of estimating the sound field generated by a limited region of turbulence in an infinitely long, straight, hard-walled pipe. The field is analysed in a co-ordinate system moving with the assumed uniform mean flow, and the possibility of eddy convection relative to that reference system is considered. Large-scale turbulence is shown to induce plane acoustic waves of intensity proportional to the sixth power of flow velocity. The same is true of small-scale turbulence of low characteristic frequency. In both cases convective effects increase the acoustic output and distribute the bulk of the energy in a mode propagating upstream against the mean flow. Small-scale turbulence of higher frequency excites more modes, the sound increasing with very nearly the eighth power of velocity ($U^{7.7}$) as soon as the second mode is excited. In the limit, when more than about 20 modes are excited, the energy output is unaffected by the constraint of the pipe walls, increasing with the eighth power of velocity, and being substantially amplified by convective motion.

1. Introduction

This paper is concerned with the sound field generated by a turbulent flow in an infinitely long straight pipe. It is chosen infinitely long so as to avoid standing waves. The motion can be analysed in a co-ordinate system moving downstream with the uniform mean flow as a sum of normal modes of oscillation, the types of mode present being determined by the cross-sectional shape and the boundary conditions. Although the field of a single mode is thus completely determined, more complex fields involving many modes will have general features independent of the particular geometry, provided the boundary conditions are of the same type. A hard-walled pipe of square cross-section is chosen here. A finite volume of fluid within the pipe is supposed to be in turbulent motion and the problem is to determine its sound field. Not all the excited modes propagate as sound. Some decay exponentially with axial distance, and we shall examine the sound field at points sufficiently far from the source that the effect of the decaying modes is negligible.

The acoustic power in the pipe can be determined from a knowledge of the pressure field and this is calculated for two types of turbulent motion. In the first, the eddies are so large that the motion is completely correlated across the pipe and all the sound is in the form of a plane wave propagating in the axial direction. The second type of motion is a statistically slowly varying flow with an eddy correlation length small compared with the cross-sectional pipe dimension.

The effects of eddy convection upstream relative to the mean flow are considered, and these are shown to augment the radiation efficiency of the turbulence. When only the plane wave mode is excited, convective effects also account for a preferential upstream radiation, but the effect tends to disappear when many acoustic modes are excited.

The analysis is quite general in the limiting cases of high and low frequencies, but for the intermediate frequency range, which, for small scale turbulence, transpires to be extensive and to encompass a wide category of subsonic flows, a more specific treatment is necessary. Then eddy convection effects are neglected and the turbulence is supposed isotropic with a particular form of cross spectral density. The example shows how the asymptotic low speed limit, when the sound energy increases as the sixth power of velocity, connects with the high frequency limit when the sound field is virtually unaffected by the presence of the constraining walls.

2. Equation for the sound field

Inviscid flow within an infinitely long straight pipe of square cross-sectional shape and side length b is considered. Rectangular co-ordinates moving with the uniform mean flow are chosen with origin on one edge of the pipe, y_1 and y_2 being co-ordinates in the cross-sectional plane, and y_3 in the axial direction.

The equation governing the sound field in the pipe is Lighthill's form of the combined continuity and momentum equations (Lighthill 1952),

$$\frac{\partial^2 \rho}{\partial \tau^2} - c^2 \nabla^2 \rho = \frac{\partial^2 T_{ij}}{\partial y_i \partial y_j}, \tag{1}$$

ρ is the density, τ is time, c is the velocity of sound and T_{ij} is the turbulence stress tensor. T_{ij} denotes the expression $(p - c^2\rho)\delta_{ij} + \rho u_i u_j$, where p is the pressure, δ_{ij} the Kronecker delta, and u_i is the particle velocity in the i-direction relative to the mean flow. It is assumed that turbulence occupies a limited region of the pipe. Outside this region where the fluctuating velocities are assumed small, the product $\rho u_i u_j$ is essentially zero. Also, for small scale adiabatic wave motion, the pressure fluctuation is balanced by $c^2\rho$ so that the turbulence stress tensor vanishes outside the turbulence.

The boundary conditions to be applied to (1) are that all waves are outgoing at infinity, and that the normal velocity vanishes at the walls. From the equation for the normal component of momentum

$$\rho \frac{\partial u_n}{\partial t} + \rho u_j \frac{\partial u_n}{\partial x_j} + \frac{\partial p}{\partial x_n} = 0,$$

the second condition is equivalent to the vanishing at the walls of the normal derivative of pressure, and, when $p = c^2\rho$, density.

Equation (1) can be solved by means of a Green's function $G(\mathbf{x}, t|\mathbf{y}, \tau)$ defined as the solution of

$$\frac{\partial^2 G}{\partial \tau^2} - c^2 \nabla^2 G = -\delta(\mathbf{x} - \mathbf{y})\,\delta(t - \tau) \tag{2}$$

with $$\frac{\partial G}{\partial n} = 0 \text{ on the boundaries.}$$

The Fourier transform of G with respect to time, written $g(\mathbf{x},\mathbf{y}|\omega)$ satisfies the equation

$$-\omega^2 g - c^2\nabla^2 g = -\delta(\mathbf{x}-\mathbf{y}) \tag{3}$$

and is related to G by the inverse transform

$$G(\mathbf{x},t|\mathbf{y},\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} g(\mathbf{x},\mathbf{y}|\omega)\exp\{-i\omega(t-\tau)\}d\omega. \tag{4}$$

The Green's function is expressed as a sum of the normal modes of oscillation, these being sines and cosines in a square pipe. Only the cosine terms are necessary here because the normal derivatives vanish at the pipe walls. The (m,n) mode is written in the form

$$\cos\frac{m\pi y_1}{b}\cos\frac{n\pi y_2}{b}f_{mn}(y_3),$$

where m and n are non-negative integers.

The function g is the sum of all such modes. $f_{mn}(y_3)$ can be determined from (3) by substitution

$$\left\{\left[\omega^2 - c^2(m^2+n^2)\frac{\pi^2}{b^2}\right]f_{mn}(y_3) + c^2\frac{\partial^2 f_{mn}}{\partial y_3^2}\right\}\cos\frac{m\pi y_1}{b}\cos\frac{n\pi y_2}{b} = \delta(\mathbf{x}-\mathbf{y}).$$

We multiply both sides of this equation by

$$\cos\frac{m\pi y_1}{b}\cos\frac{m\pi y_2}{b}$$

and integrate over the cross-sectional plane to produce a reduced equation

$$c^2\frac{\partial^2 f_{mn}}{\partial y_3^2} + c^2 k_{mn}^2 f_{mn} = C_{mn}(x_1,x_2)\,\delta(x_3-y_3), \tag{5}$$

where
$$\left.\begin{aligned} k_{mn}^2 &= \frac{\omega^2}{c^2} - (m^2+n^2)\frac{\pi^2}{b^2}, \\ C_{mn}(x_1,x_2) &= \frac{1}{b^2\epsilon_n\epsilon_m}\cos\frac{m\pi x_1}{b}\cos\frac{n\pi x_2}{b}, \end{aligned}\right\} \tag{6}$$

and
$$\begin{aligned} \epsilon_n &= 1 \quad \text{if} \quad n = 0, \\ &= \tfrac{1}{2} \quad \text{if} \quad n > 0. \end{aligned}$$

A solution of (5) with outgoing waves at infinity is (Morse & Feshbach 1953, p. 810)

$$f_{mn} = -\frac{1}{2c^2}C_{mn}\frac{\exp(ik_{mn}|x_3-y_3|)}{k_{mn}}. \tag{7}$$

By summing all these terms to produce g, and then inverting the transform according to (4), we obtain the Green's function G satisfying (2) and hard-walled boundary conditions,

$$G(\mathbf{x},t|\mathbf{y},\tau) = -\frac{i}{4\pi c^2}\int_{-\infty}^{\infty}\left[\sum_{m,n=0}^{\infty} C_{mn}(x_1,x_2)\cos\frac{m\pi y_1}{b}\cos\frac{n\pi y_2}{b}\frac{\exp(ik_{mn}|x_3-y_3|)}{k_{mn}}\right] \times \exp\{-i\omega(t-\tau)\}d\omega. \tag{8}$$

We can now superpose the elementary solutions G and express the formal solution of (1) for the density perturbation in the pipe in the form

$$\int_v \int_t G(\mathbf{x},t|\mathbf{y},\tau)\frac{\partial^2 T_{ij}}{\partial y_i \partial y_j}(\mathbf{y},\tau)\,d\mathbf{y}\,d\tau,$$

where the volume integral is over all space and the time integral ranges from $-\infty$ to $+\infty$. The Green's function chosen is an exact form, and, with the boundary conditions specified, ensures that the solution involves no surface integrals. Thus

$$p(\mathbf{x},t) = c^2\rho(\mathbf{x},t) = \frac{-i}{4\pi}\sum_{m,n=0}^{\infty} C_{mn}(x_1,x_2)$$
$$\times \int_v d\mathbf{y}\int_{-\infty}^{\infty} d\tau \int_{-\infty}^{\infty} d\omega \frac{\partial^2 T_{ij}}{\partial y_i y_j}\cos\frac{m\pi y_1}{b}\cos\frac{n\pi y_2}{b}\frac{\exp\{ik_{mn}|x_3-y_3|-i\omega(t-\tau)\}}{k_{mn}}. \quad (9)$$

For large positive values of x_3, the exponential term is

$$\exp\{ik_{mn}(x_3-y_3)-i\omega(t-\tau)\}.$$

The term k_{mn}, defined by (6), denotes the wave-number in the axial direction. For propagating wave motion this must be real, which it is above the cut off frequency which we denote by ω_{mn}

$$\omega_{mn}^2 = (m^2+n^2)\frac{\pi^2c^2}{b^2}. \quad (10)$$

For frequencies less than the cut-off frequency, k_{mn}^2 is negative and the pressure in that mode decays exponentially away from the source. It is convenient to split the ω integration in (9) into a propagating and a non-propagating part,

$$\int_{-\infty}^{\infty} d\omega \equiv \left\{\int_{-\infty}^{-\omega_{mn}} d\omega + \int_{\omega_{mn}}^{\infty} d\omega\right\} + \int_{-\omega_{mn}}^{\omega_{mn}} d\omega.$$

In the range $\omega^2 < \omega_{mn}^2$, k_{mn} is imaginary. At large distances from the source, this part of the ω integration can be ignored. The remaining range corresponds to all real k_{nm}, so that the propagating part of the ω integration in (9) can be written in the form

$$\int_{-\infty}^{-\omega_{mn}} d\omega + \int_{\omega_{mn}}^{\infty} d\omega \equiv \int_{-\infty}^{\infty} dk_{mn}\frac{k_{mn}c^2}{\omega}. \quad (11)$$

A physical interpretation of the pressure field becomes apparent if the sound pressure $p_{mn}(\mathbf{x},\omega)$ in a particular mode at frequency ω is considered. The τ integration in (9) can be performed and the resulting Fourier transform of the source field $T_{ij}(\mathbf{y},\tau)$ is denoted by $\theta_{ij}(\mathbf{y},\omega)$. $p_{mn}(\omega)$ is thus defined by the equation

$$p_{mn}(\mathbf{x},\omega) = -\frac{iC_{mn}}{2}\frac{\exp(ik_{mn}x_3)}{k_{mn}}\int_v d\mathbf{y}\frac{\partial^2\theta_{ij}}{\partial y_i \partial y_j}(\mathbf{y},\omega)\cos\frac{m\pi y_1}{b}\cos\frac{n\pi y_2}{b}$$
$$\times \exp(-ik_{mn}y_3). \quad (12)$$

This expression for p_{mn} is the sum of four rays of plane sound waves, each ray having a sound pressure $p_{mn}^{\phi}(\mathbf{x},\omega)$, where ϕ, the number of the ray equals 1, 2, 3 or 4. This is made clear by expressing the cosine terms as the sum of four exponentials. The case, $\phi = 1$ is chosen to correspond to the exponential term

$$\frac{1}{4}\exp\left\{-\left(\frac{im\pi}{b}y_1+\frac{in\pi}{b}y_2+ik_{mn}y_3\right)\right\}$$

in the integrand. The other rays correspond to other combinations of $\pm m$ and $\pm n$. If the $\mathbf{y}$ axes are rotated to new axes $\boldsymbol{\eta}'$, where η_3' makes angles $\cos^{-1}(m\pi c/b\omega)$ and $\cos^{-1}(n\pi c/b\omega)$ with the y_1 and y_2 axes respectively (the η_1' and η_2' axes can be chosen arbitrarily) the exponential in the $\phi = 1$ ray is $\exp(-i\eta_3'\omega/c)$ and the pressure p'_{mn} is a plane wave travelling with velocity c along the η_3' axis. Thus the strength of each ray is that generated by the turbulence in an unbounded space, and this is easily computed by the usual techniques of aerodynamic noise theory. We shall postpone this computation until we have found a statistical mean value, but we shall make use of the fact that a particular mode can be identified quite simply with a particular ray that would be generated were the turbulence not contained in the pipe.

3. Acoustic power

The acoustic power, or the rate at which energy is propagating down the pipe at a particular frequency ω, is the integral over the cross-section of the downstream component of the intensity vector, $\overline{p\mathbf{u}}$. This we denote by $P(\omega)$,

$$P(\omega) = E\int_s \overline{p(\mathbf{x},\omega)u_3(\mathbf{x},-\omega)}\,dx_1dx_2, \tag{13}$$

where u_3 is the axial particle velocity of the disturbance and the bar signifies a mean value. E is the small normalizing constant that enters when computing a spectrum level from the product of Fourier transforms. u_3 has a similar form to the pressure, p, and is determined from the linearized momentum equation

$$\frac{\partial p}{\partial x_3}+\rho\frac{\partial u_3}{\partial t} = 0.$$

In Fourier components, this becomes

$$u_3 = \frac{k_{mn}}{\omega\rho}p. \tag{14}$$

The integration over the cross-sectional plane ensures that the correlation over different modes is zero, and that the total power carried by the flow is the sum of the power in each mode. The power in the (m,n) mode, $P_{mn}(\omega)$ is obtained from (12) to (14). In the expression for the complex conjugate of the velocity transform, the frequency is denoted by $-\omega$, and the negative sign of the square root is to be selected when writing the wave-number k_{mn}. The sound power in a mode then has the form;

$$P_{mn}(\omega) = \frac{1}{4b^2\epsilon_n\epsilon_m}\int_v\int_v dy\,dz\,E\,\overline{\frac{\partial^2\theta_{ij}}{\partial y_i\partial y_j}(\mathbf{y},\omega)\frac{\partial^2\theta_{kl}}{\partial z_k\partial z_l}(\mathbf{z},-\omega)}$$
$$\times\cos\frac{m\pi y_1}{b}\cos\frac{n\pi y_2}{b}\cos\frac{m\pi z_1}{b}\cos\frac{n\pi z_2}{b}\frac{\exp\{-ik_{mn}(y_3-z_3)\}}{k_{mn}\omega\rho}. \tag{15}$$

The sound power is seen to be constant as is to be expected in a hard-walled pipe where energy is conserved.

It is assumed that the source field is locally homogeneous so that the correlation function is a function only of the displacement $\boldsymbol{\lambda} = \mathbf{z} - \mathbf{y}$. The source being considered is a double divergence, so that the correlation function is a quadruple divergence of the correlation function of the stress tensor, R_{ijkl},

$$R_{ijkl}(\boldsymbol{\lambda}, \omega) = \overline{E\theta_{ij}(\mathbf{y}, \omega)\theta_{kl}(\mathbf{y}+\boldsymbol{\lambda}, -\omega)}$$

$$E\,\overline{\frac{\partial^2\theta_{ij}}{\partial y_i \partial y_j}(\mathbf{y}, \omega)\frac{\partial^2\theta_{kl}}{\partial z_k \partial z_l}(\mathbf{z}, -\omega)} = \frac{\partial^4}{\partial\lambda_i\partial\lambda_j\partial\lambda_k\partial\lambda_l}R_{ijkl}(\boldsymbol{\lambda}, \omega). \tag{16}$$

Equation (15) for the sound power in a mode now takes the form,

$$P_{mn}(\omega) = \frac{1}{4b^2\epsilon_n\epsilon_m}\int_v\int_v dy\,d\boldsymbol{\lambda}\frac{\partial^4}{\partial\lambda_i\partial\lambda_j\partial\lambda_k\partial\lambda_l}R_{ijkl}(\boldsymbol{\lambda}, \omega)$$
$$\times \cos\frac{m\pi y_1}{b}\cos\frac{n\pi y_2}{b}\cos\frac{m\pi z_1}{b}\cos\frac{n\pi z_2}{b}\frac{e^{-ik_{mn}\lambda_3}}{k_{mn}\omega\rho}. \tag{17}$$

In general, the limits on the $\boldsymbol{\lambda}$ integration are functions of $\mathbf{y}$. However, the integration is straightforward in two particular limiting cases and these we will now consider.

In the first case, the eddies in the pipe are so large that the motion is completely correlated across the pipe. Thus, R_{ijkl} is constant in the cross-sectional plane and the integration in the λ_1 and λ_2 variables is over the cosine terms. The integral is non-zero only when $m = n = 0$. Thus the only sound generated is in the form of a plane wave propagating in the axial direction with velocity c. Only one element of the correlation tensor yields a non-vanishing integral in this case, and that corresponds to the longitudinal quadrupoles with axis parallel to the pipe axis. The acoustic power generated by large-scale turbulent motion is thus very simply described as

$$P_{00}(\omega) = \frac{V}{4b^2}\frac{\omega^2}{c^2}\frac{1}{\rho c}b^2\int_{-\infty}^{\infty} R_{3333}(\lambda_3, \omega)\exp\left(-i\frac{\omega}{c}\lambda_3\right)d\lambda_3, \tag{18}$$

where V is written for the volume occupied by the turbulence. The total power P, is

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} P_{00}(\omega)\,d\omega,$$

and the integration can be absorbed in producing the cross-correlation function

$$R_{3333}(\lambda_3, \tau) = \overline{T_{33}(0,0)T_{33}(\lambda_3, \tau)},$$

$$P = \frac{1}{2\pi}\int_{-\infty}^{\infty} P_{00}(\omega)\,d\omega = \frac{V}{4c^3\rho}\int_{-\infty}^{\infty}\frac{\partial^2}{\partial\tau^2}R_{3333}(\lambda_3, \tau)\bigg|_{\tau=\lambda_3/c} d\lambda_3. \tag{19}$$

Now if the turbulence and the turbulent region is moving at Mach number M_c upstream, relative to the moving uniform fluid far downstream in the pipe, the correlation function is more naturally specified in terms of a moving co-ordinate system, $\lambda_3^m = \lambda_3 + M_c c\tau$. The moving axis turbulence correlation tensor $R^m(\lambda_3^m, \tau) = R(\lambda_3, \tau)$, then enters the equation which can be written in a moving axis form through Lighthill's transformation (Lighthill 1952),

$$P = \frac{V}{4c^3\rho}\frac{1}{|1+M_c|^4}\int_{-\infty}^{\infty}\frac{\partial^2}{\partial\tau^2}R^m_{3333}(\lambda^m_3,\tau)\bigg|_{\tau=\lambda_3/\{(1+M_c)c\}} d\lambda^m_3. \tag{20}$$

The sound power moving upstream is given by precisely the same expression with M_c replaced by $-M_c$, so that the ratio of downstream power to upstream power is

$$\left|\frac{1-M_c}{1+M_c}\right|^4$$

provided retarded time effects are negligible.

The dimensional form of the power can be obtained by setting αb to correspond to a turbulence length scale and $\beta U/b$ to a frequency. R^m_{3333} will be proportional to $\sigma^4\rho^2U^4$, σ being the ratio of the characteristic turbulent velocity to the mean velocity, so that

$$P \sim \frac{V}{4|1\pm M_c|^4 b}\rho U^3M^3\sigma^4\alpha\beta^2. \tag{21}$$

The + and − sign correspond to downstream and upstream conditions respectively, and M has been written for the flow Mach number U/c.

This exceeds the power radiated by the same turbulence in free space by a factor that includes M^{-2}, so that the restraining influence of the walls considerably augments the radiation efficiency of large-scale turbulence. This is precisely the conclusion one would make by applying Curle's (1955) dimensional analysis to the model and regarding the problem as a three-dimensional one in the presence of rigid boundaries.

The second limiting type of turbulent motion we consider has eddies small compared to the pipe dimension. The tubulence is assumed slowly varying so that it can be treated as locally homogeneous. For sufficiently small correlation lengths, the limits on the $\boldsymbol{\lambda}$ integration in (17) can be taken as $\pm\infty$, and the $\mathbf{y}$ and $\boldsymbol{\lambda}$ integrations performed separately. Edge effects from the flow near the wall are negligible. This is because the source function T_{ij} vanishes at the walls and all eddies must have zero strength there. The volume of eddies near the walls compared with the total volume of the source region is of the order $\alpha b^2/b^2$, where αb is the effective correlation length. This is negligible for sufficiently small α.

The y integration can now be performed independently in (17), and is affected by simply replacing the integral by V, homogeneous turbulent flow being assumed to exist over a source volume V,

$$P_{mn}(\omega) = \frac{V}{4b^2}\int_{-\infty}^{\infty}\iint d\boldsymbol{\lambda}\frac{\partial^4}{\partial\lambda_i\partial\lambda_j\partial\lambda_k\partial\lambda_l}R_{ijkl}(\boldsymbol{\lambda},\omega) \times\cos\frac{m\pi\lambda_1}{b}\cos\frac{n\pi\lambda_2}{b}\frac{\exp(-ik_{mn}\lambda_3)}{k_{mn}\omega\rho}. \tag{22}$$

The cosine terms can again be expanded into four exponential terms, and the sound power in the pipe expressed as the sum of the sound power along four rays. Again we label the rays by $\phi = 1, 2, 3, 4$ and call the new co-ordinates obtained by simple rotation of the $\boldsymbol{\lambda}$ system $\boldsymbol{\eta}^{\phi}$. Equation (22) can then be written in terms of the inclined co-ordinate systems as the sum of the power in the four rays,

$$P_{mn}(\omega) = \frac{V}{16b^2k_{mn}\omega\rho}\sum_{\phi=1}^{4}\int_{-\infty}^{\infty}\iint d\boldsymbol{\eta}^{\phi}\frac{\partial^4}{\partial\eta_i^{\phi}\partial\eta_j^{\phi}\partial\eta_k^{\phi}\partial\eta_l^{\phi}}R_{ijkl}^{\phi}(\eta^{\phi},\omega)\exp\left(-i\frac{\omega}{c}\eta_3^{\phi}\right). \quad (23)$$

$R_{ijkl}^{\phi}(\boldsymbol{\eta}^{\phi},\omega) = R_{ijkl}(\boldsymbol{\lambda},\omega)$ is the correlation tensor expressed as a function of the inclined co-ordinate system, and η_3^{ϕ} is the direction parallel to the ray propagation direction.

The limiting cases of this result are simply evaluated. In the first case, if the characteristic frequencies are low (i.e. $\omega \ll \pi c/b$) only the zero order mode can propagate and (23) can be transformed by integration by parts to a form almost identical with (18),

$$P_{00}(\omega) = \frac{V}{4b^2}\frac{\omega^2}{c^2}\frac{1}{\rho c}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}R_{3333}(\boldsymbol{\lambda},\omega)\exp\left(-i\frac{\omega}{c}\lambda_3\right)d\lambda_1 d\lambda_2 d\lambda_3. \quad (24)$$

The integration over λ_1 and λ_2 yields a multiplier $(\alpha b)^2$, so that the sound of the small-scale turbulence at low enough frequency is seen, by comparison with (18), to be precisely α^2 times the sound generated by very large-scale turbulence. Consequently we can obtain the dimensional trends on the power variation by multiplying (21) by α^2,

$$P \sim \frac{V}{4|1\pm M_c|^4 b}\rho U^3 M^3\sigma^4\alpha^3\beta^2. \quad (25)$$

Again the + and − signs refer to the power propagating downstream and upstream respectively.

The second limiting case is when the frequencies are high enough relative to $\pi c/b$, that all modes propagate. Then the total power travelling down the pipe is exactly equal to the total power that would be radiated into the downstream half space were the turbulence radiating in a uniform acoustic medium in the absence of the pipe. This can be shown as follows.

Provided the modes are sufficiently numerous, the summation can be viewed as an integral over a continuous modal distribution of modal density unity. A doubly infinite range of both m and n is necesssary, the positive and negative parts corresponding to the various rays we designate by ϕ. The total power propagating down the pipe is equal to the integral of the power spectral density over frequency, and from (23) can be written,

$$\begin{aligned}P &= \frac{1}{2\pi}\int_{-\infty}^{\infty}d\omega\int_{-\infty}^{\infty}dm\int_{-\infty}^{\infty}dn\,P_{mn}(\omega)\\ &= \iint\frac{V}{16b^2k_{mn}\omega\rho}\int_V\int_{-\infty}^{\infty}\frac{\partial^4}{\partial\eta_i^{\phi}\partial\eta_j^{\phi}\partial\eta_k^{\phi}\partial\eta_l^{\phi}}R_{ijkl}^{\phi}(\boldsymbol{\eta}^{\phi},\omega)\exp\left(-i\frac{\omega}{c}\eta_3^{\phi}\right)\frac{d\omega}{2\pi}dm\,dn\,d\boldsymbol{\eta}^{\phi}.\end{aligned} \quad (26)$$

The frequency integral generates in the integrand a quadruple divergence of the stress tensor cross-correlation at the retarded time associated with wave propagation in the direction $\hat{\boldsymbol{\phi}}$,

$$P = \iint\frac{V}{16b^2k_{mn}\omega\rho}\int_{V(\eta)}\frac{\partial^4}{\partial\eta_i^{\phi}\partial\eta_j^{\phi}\partial\eta_k^{\phi}\partial\eta_l^{\phi}}R_{ijkl}^{\phi}\left(\boldsymbol{\eta}^{\phi},\frac{\eta_3^{\phi}}{c}\right)d\boldsymbol{\eta}^{\phi}dm\,dn. \quad (27)$$

Now the intensity $I(r, \hat{\boldsymbol{\phi}})$ that would arise at large distance r and direction $\hat{\boldsymbol{\phi}}$, were the turbulence radiating in free space, is

$$I(r, \hat{\boldsymbol{\phi}}) = \frac{V}{16\pi^2 r^2 \rho c} \int_{V(\eta)} \frac{\partial^4}{\partial\eta_i^\phi \partial\eta_j^\phi \partial\eta_k^\phi \partial\eta_l^\phi} R^\phi_{ijkl}\left(\eta^\phi, \frac{\eta_3^\phi}{c}\right) d\eta^\phi. \tag{28}$$

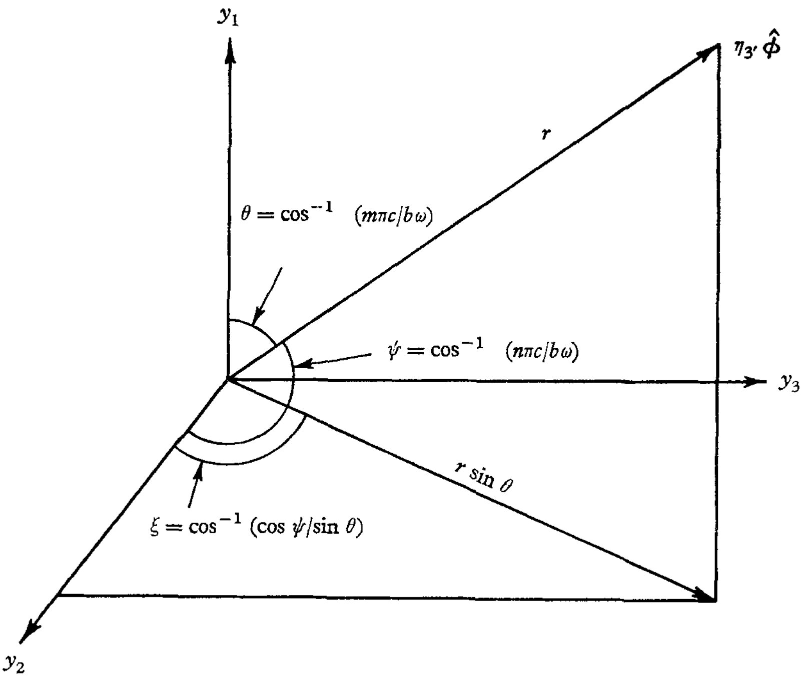

FIGURE 1. Diagram illustrating the co-ordinate systems and the ray direction $\hat{\boldsymbol{\phi}}$. $\hat{\boldsymbol{\phi}}$ ranges over the $+\text{ve}\, y_3$ half space as m and n vary over $(-\infty, \infty)$.

Consequently, the power radiated down the pipe can be expressed directly in terms of the hypothetical intensity field that would be generated by the turbulence in free space

$$P = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{\pi^2 c r^2}{b^2 \omega k_{mn}} I(r, \hat{\boldsymbol{\phi}})\, dm\, dn. \tag{29}$$

The direction $\hat{\boldsymbol{\phi}}$ is determined by m and n, as is illustrated in figure 1, and an area element over the surface of a sphere at constant r is $r^2 \sin\theta\, d\theta\, d\xi$. This in turn is equal to

$$\frac{\pi^2 c}{b^2 \omega k_{mn}} r^2\, dm\, dn,$$

so that the integrals in (29) can be written as a surface integral over the downstream $(+\text{ve}\, y_3)$ hemispherical surface,

$$P = \int_{s(+\text{ve}\, y_3)} I(r, \hat{\boldsymbol{\phi}})\, ds. \tag{30}$$

This surface integral of the outward component of the intensity vector is the total power radiated by the turbulence into the downstream half space, so that the equality is a statement that the total power generated by the turbulence in a pipe is completely unaffected by the presence of the pipe, provided that the frequencies are sufficiently high relative to $\pi c/b$. This statement is true whether or not there are convective effects, so that the effects of source motion can be carried over from the three-dimensional problem. Since the convective effects vary with ray direction, they depend on the quadrupole type, each type giving different emphasis to the directions coincident with the quadrupole axes. In isotropic turbulence Lighthill (1952) showed that convective effects increase the radiated power by a factor,

$$\frac{1+2M_c^2+M_c^4/5}{(1-M_c^2)^5}.$$

In this case the radiation is more or less evenly split into parts propagating up and down the pipe, though relative to the fluid the majority radiates upstream. This is because only a limited number of upstream propagating rays have an axial propagation speed in excess of the flow velocity and can avoid being swept downstream with the flow.

In this high frequency limit, a dimensional analysis applied to (28) and (30), together with the known effects of source convection shows that the total radiated power varies like

$$P \sim \frac{1+2M_c^2+\frac{1}{5}M_c^4}{(1-M_c^2)^5}\frac{V}{4b}\rho U^3 M^5 \sigma^4 \alpha^3 \beta^4. \tag{31}$$

At intermediate frequencies the field is very much a function of the precise number of modes contributing to the radiation. It is much simpler to discard the effects of source convection in that case and to regard the turbulence as moving with the stream. We recognize that the integral in (23) is simply a Fourier transform operation which produces the power spectral density tensor $H_{ijkl}(\omega/c\hat{\boldsymbol{\phi}}, \omega)$ (Ffowcs Williams 1963), so that (23) can formally be written as

$$P_{mn}(\omega) = \frac{V}{16b^2 k_{mn}\omega\rho}\sum_{\phi=1}^{4}\left(\frac{\omega}{c}\right)^4 l_i^\phi l_j^\phi l_k^\phi l_l^\phi H_{ijkl}\left(\frac{\omega}{c}\hat{\boldsymbol{\phi}}, \omega\right), \tag{32}$$

l_i^ϕ being the direction cosine of the i-direction in the Cartesian set formed on the pipe axes onto the ϕ ray, a ray parallel to the unit vector $\hat{\boldsymbol{\phi}}$.

Certain assumptions can now be made about the form of the power spectral density. H is, by definition, an even function of wave-number. We shall assume that it is also an even function of each component of wave-number in a Cartesian set based on the pipe. This assumption does not seem unreasonable, implying that if principal axes of correlation exist, they are coincident with the Cartesian axes of the pipe. When this is so, the summation over the four rays ensures that the result is zero unless i, j, k and l are equal in pairs.

$P_{mn}(\omega)$ is the power generated in one mode at frequency ω by a source distribution of quadrupoles. The dimensional form of P_{mn} depends on the type of quadrupoles. H_{ijkl} is a four-dimensional Fourier transform and is of the order of magnitude of

$$\sigma^4(\rho U^2)^2\frac{\alpha^3 b^4}{\beta U}.$$

From (32), $P_{mn}(\omega)$, thus has an order of magnitude given by the proportionality,

$$P_{mn} \sim \frac{\sigma^4}{16}\rho b^2 V \frac{\alpha^3}{\beta} U^2 M \frac{\omega^2}{c^2} (l_i^\phi l_j^\phi l_k^\phi l_l^\phi) f_{ijkl}, \tag{33}$$

f_{ijkl} being a non-dimensional tensor characterizing the type of quadrupoles present in the pipe. k_{mn} has been set proportional to ω/c. This order of magnitude of the source is modified by the directional term $l_i^\phi l_j^\phi l_k^\phi l_l^\phi$. For a quadrupole with an axis in the cross-sectional plane, the direction cosine is either $m\pi c/\omega b$ or $n\pi c/\omega b$. For a quadrupole with an axis parallel to the pipe axis, it is $k_{mn} c/\omega$, which we take as unity. Because i, j, k and l, must be equal in pairs, the directional factor thus introduces into (33) a factor

$$\frac{c^4}{b^4\omega^4}, \quad \frac{c^2}{b^2\omega^2} \quad \text{or} \quad 1,$$

respectively, depending as the quadrupole has axes wholely in the cross-sectional plane, say T_{11}; is a lateral quadrupole T_{ij} with one of either i or j equal to 3; or is *a* T_{33} quadrupole with axes parallel to the pipe axis.

This directional attenuation with increasing frequency is easily explained using the expansion into rays of plane waves. In a given mode the angle made by the direction of propagation of the rays and the pipe axis decreases as the frequency increases. Thus the direction of sound propagation makes a larger angle with a quadrupole axis in the cross-sectional plane, so decreasing the efficiency of propagation.

If the mean flow velocity dependence of the frequency is introduced the directional effect is masked as it is overwhelmed by the increase in source efficiency. P_{mn} is then of the order of magnitude of

$$\frac{\sigma^4}{16}\rho U^2 M^{-1} V \frac{\alpha^3}{\beta^3}, \quad \frac{\alpha^4}{16}\rho U^2 M V \frac{\alpha^3}{\beta} \quad \text{or} \quad \frac{\sigma^4}{16}\rho U^2 M^3 V \alpha^3 \beta,$$

respectively, for the three types of quadrupoles considered above. For a specific increase in velocity, the increase in the magnitude of the sound generated in one mode is then greatest for a T_{33} quadrupole.

4. Sound power from isotropic turbulence

In isotropic turbulence, the sound power has no preferred direction and the source spectral function in (32) is a function of the modulus of the wave vector and frequency only. If αb and $(\beta U/b)^{-1}$, respectively, again denote the typical turbulence length and time scales, a suitable form for the spectrum is

$$l_i^\phi l_j^\phi l_k^\phi l_l^\phi H_{ijkl}\left(\frac{\omega}{c}\hat{\boldsymbol{\phi}}, \omega\right) = \sigma^4 \rho^2 U^3 \frac{\alpha^3 b^4}{\beta} \exp\left\{-\frac{1}{4\pi}\left(\frac{\omega}{c}\right)^2 \alpha^2 b^2\right\} \exp\left(-\frac{\omega^2}{4\pi}\frac{b^2}{\beta^2 U^2}\right). \tag{34}$$

The exponential terms have, respectively, a wave-number and frequency dependence. At sufficiently low Mach numbers, the frequency term in the exponential dominates, and will indeed be considerably larger than the wave-number term at all subsonic speeds. In shear flow turbulence, $\beta \sim \frac{1}{5}\alpha$ (Davies, Fisher & Barratt 1963). In pipe flow β will, in general, be smaller since the eddies have a longer lifetime, and will always be such that the important dependence of the spectrum function is on frequency. The spectrum function can consequently be written as

$$\sigma^4\rho^2U^3\frac{\alpha^3}{\beta}b^4\exp\left\{-\frac{1}{4\pi}\left(\frac{\omega b}{c}\right)^2\frac{1}{\beta^2M^2}\right\}.$$

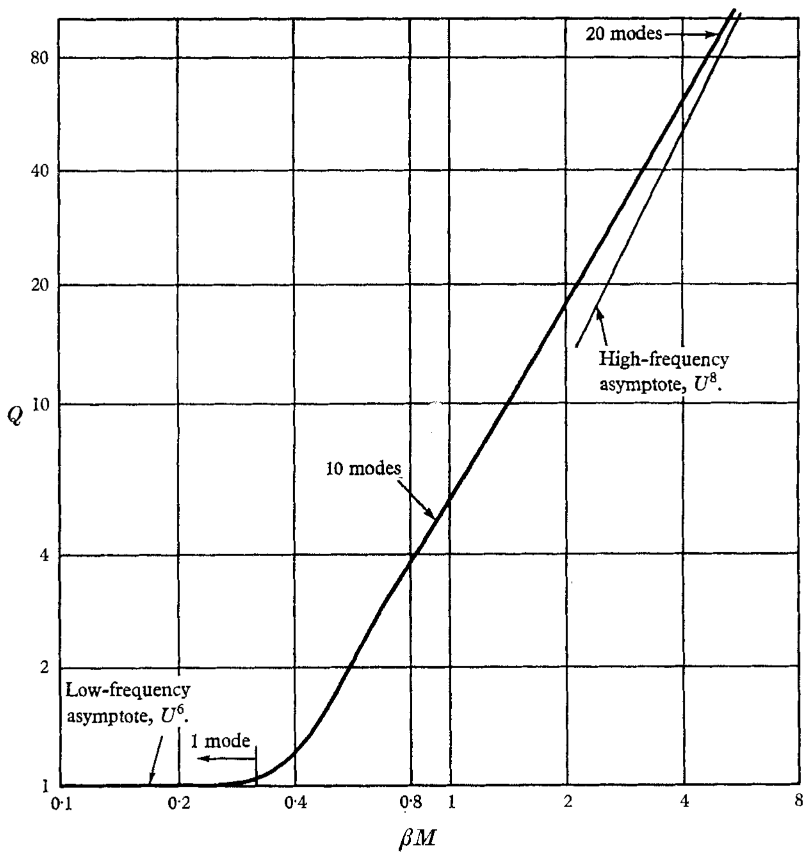

FIGURE 2. Diagram of Q versus βM together with the number of modes needed for 99% accuracy.

The total power generated is found by integrating (32) for $1/(2\pi)\,P_{mn}(\omega)$ over all frequencies and summing the power in all modes. For isotropic turbulence of the form chosen, and by making use of expression (11) for the change of variable in the integration, the total power P is

$$P = \frac{\sigma^4}{32\pi}\rho b^2VU^3\frac{\alpha^3}{\beta}\sum_{m,\,n=0}^{\infty}\exp\left\{-(m^2+n^2)\frac{\pi}{4\beta^2M^2}\right\}$$
$$\times\int_{-\infty}^{\infty}dk_{mn}[k_{mn}^2+(m^2+n^2)\pi^2/b^2]\exp\left(-\frac{k_{mn}^2b^2}{4\pi\beta^2M^2}\right).$$

The integration is straightforward and leads to the equation

$$P = \frac{\pi}{8}\sigma^4\frac{\alpha^3\beta^2V\rho U^3M^3}{b}\sum_{m,\,n=0}^{\infty}\left[1+(m^2+n^2)\frac{\pi}{2\beta^2M^2}\right]\exp\left\{-(m^2+n^2)\frac{\pi}{4\beta^2M^2}\right\}. \quad (35)$$

At low Mach numbers, or low typical frequencies, βM is small and

$$\exp\left\{-(m^2+n^2)\frac{\pi}{4\beta^2M^2}\right\}$$

is negligible whenever $(m^2+n^2) \neq 0$. Then only the first, or plane wave mode is significantly excited and the sound power increases as the sixth power of flow velocity, being given explicitly by the expression

$$P|_{\beta^2M^2\ll 1} = \frac{\pi}{8}\sigma^4\alpha^3\beta^2\frac{V}{b}\rho U^3M^3, \tag{36}$$

which compares directly with (25).

On the other hand, at high Mach number, or high typical frequencies, many modes contribute to the sum and the summation can be approximated by an integral over the modes, the modal density being unity. The sound power then increases with the eighth power of flow velocity, being equal to

$$P|_{\beta^2M^2\gg 1} = \frac{3\pi}{8}\sigma^4\alpha^3\beta^4\frac{V}{b}\rho U^3M^5, \tag{37}$$

and this compares directly with (31).

The exact summation together with these limiting forms is illustrated in figure 2. This is a plot of Q versus βM, Q being defined from (35) as

$$Q = \frac{8P}{\pi\sigma^4\alpha^3\beta^2V\rho U^3M^3/b},$$

$$Q = \sum_{m,\,n=0}^{\infty}\sum\left[1+(m^2+n^2)\frac{\pi}{2\beta^2M^2}\right]\exp\left\{-(m^2+n^2)\frac{\pi}{4\beta^2M^2}\right\}. \tag{38}$$

The number of modes needed to represent the answer within 1 % is also indicated in figure 2, from which it can be seen that the summation is effectively complete when twenty modes become active. Thereafter the high frequency limit of (31) and (37) is applicable. This evidently becomes a good approximation whenever βM exceeds a value of about eight though the variation of sound power with velocity is very close to U^8 above $\beta M = 0{\cdot}4$. It is so close in fact that the error incurred by assuming the asymptotic forms to hold never exceeds a factor of 2. Below a value of $\beta M = 0{\cdot}3$, only the first mode is important and that is the range treated by (25) and (36).

5. Conclusions

The containment of turbulent flow within a hard-walled pipe has a marked effect on its radiation efficiency. The sound generated by large-scale turbulence, highly correlated across the pipe, is in a form of a plane wave, and increases with the sixth power of flow velocity. It thus exceeds the energy radiated by the same turbulence in free space by a factor M^{-2}. The same is true of the sound field generated by small-scale turbulence of low frequency when only the plane wave propagates, all other modes decaying exponentially with distance. In both these plane wave situations, if the turbulence is moving upstream relative to the flow, as it would be if generated by flow past fixed obstructions, convective effects result in a large increase in the acoustic power output, together with a strong tendency for most of the sound to propagate upstream against the flow. Small-scale turbulence of higher frequency excites an increasing number of modes until at very high frequency effectively all modes are excited and the turbulence radiates in precisely the same way as it would in free space. Again convective effects increase the power output very substantially, but in this case the energy is approximately evenly divided into downstream and upstream travelling components. Again, relative to the flow, the majority of the field propagates upstream, but not all of it rapidly enough to avoid being convected downstream by the mean flow. A detailed computation of the field radiated by small-scale isotropic turbulence that moves with the flow indicates that the asymptotic high frequency result is effectively established when about 20 modes are significantly excited, and this occurs when the characteristic frequency of the turbulence is about ten times the first cut off frequency of the pipe. On the other hand an assumption that the asymptotic form is established immediately the second mode supports a significant fraction of the energy, that is at $\beta M = 0{\cdot}4$, results in a very small error, being an underestimate of the actual power by a factor always less than 3 db. It is difficult to speculate on the magnitude of the scale coefficients α and β because they have not yet been measured in pipe flows. However, an order of magnitude estimate might be obtained by taking their relative measure as equal to that found in shear flow turbulence where $\beta \sim \frac{1}{5}\alpha$. Then, if the eddy dimensions are about $\frac{1}{10}$th of the pipe width, $\alpha = 10^{-1}$, $\beta \sim 2$. In that case it can be seen from figure 2 that an eighth power dependence on flow velocity is effectively established above a flow Mach number of 0·2 below which the sound power varies as the sixth power of velocity.

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